Free motion on the Poisson plane and sphere

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Abstract

Poisson plane and sphere — homogeneous spaces of Poisson groups $E(2)$ and $SU(2)$ (resp.) — have phase spaces (corresponding symplectic groupoids), in which a free Hamiltonian is naturally defined. We solve the equations of motion and point out some unexpected features: free motion on the plane is bounded (periodic) and free trajectories on the sphere are all circles except the big ones.

1 Introduction

This paper is a continuation of an earlier work [1, 2, 3, 4, 5] on examples of classical mechanical systems based on Poisson symmetry. Our examples have the following common structure: configurations are described by a Poisson manifold Q which is a quotient of a Poisson group G by a Poisson subgroup H (we shall assume that both groups are connected). In such a case of a Poisson homogeneous manifold, it is particularly simple to describe its phase space $\mathbf{Ph}\,Q$ and the canonical moment map $J\colon \mathbf{Ph}\,Q \to \overline{G^*}$, where G^* is the Poisson dual of G (for any Poisson manifold P, we denote by $\overline{P}$ the same manifold with the opposite Poisson structure). $\mathbf{Ph}\,Q$ can be identified with the symplectic reduction of $\mathbf{Ph}\,G$ with respect to the constraint

$$J_H^{\mathrm{right}} = \{e\}, \tag{1}$$

where $J_H^{\mathrm{right}} : \mathbf{Ph}\,G \to H^*$ is the canonical moment for the (phase lift of the) right translations on G by elements of H, and e is the unit of H^*. Then J is just the restriction of the canonical moment for the (phase lift of the) left translations on G to the above constraint (J is constant on the characteristics because left and right translations commute).

In favorable cases, $\mathbf{Ph}\,G = G \bowtie G^* \cong G \cdot G^*$ is just the Manin group [6] (the group corresponding to the Lie algebra $\mathfrak{g} \bowtie \mathfrak{g}^*$ of the Manin triple). The (phase lifts of the) left and the right translations on G have canonical moment

$$J^{\mathrm{left}}(g \cdot g^*) = {}^{g}g^* \qquad \text{and} \qquad J^{\mathrm{right}}(g \cdot g^*) = g^*, \tag{2}$$

respectively. Here $g \in G$, $g^* \in G^*$, and ${}^{g}g^* \in G^*$ is the *dressing action* of g on g^* defined by

$$g \cdot g^* = {}^{g}g^* \cdot g',$$

where $g' \in G$ (g' is denoted by g^{g^*} in [6]). To the (Poisson) inclusion of a Poisson subgroup H in G there corresponds a Poisson projection from G^* to H^*. The kernel of this projection is the subgroup H° in G^* corresponding to the annihilator $\mathfrak{h}^\circ$ of $\mathfrak{h}$ in G^*. Composing J^{right} with this projection we obtain J_H^{right}, hence the constraint submanifold in $G \cdot G^*$ defined by (1) is simply

$$K = G \cdot H^\circ. \tag{3}$$

The phase lift of a right translation by $h \in H$ on G is the right translation by h on $G \cdot G^*$. Therefore, the symplectic reduction of K which coincides with the quotient of K by the action of H is just the set of the right H-cosets in K,

$$\mathbf{Ph}\, Q = (G \cdot H^\circ)/H, \tag{4}$$

which is a bundle over $Q = G/H$, associated with the principal H-bundle G over G/H and the action of H on H° (it is easy to see that $[\mathfrak{h}, \mathfrak{h}^\circ] \subset \mathfrak{h}^\circ$ in the Lie algebra of the Manin triple, hence $h^{-1}H^\circ h \subset H^\circ$ for $h \in H$ and therefore $G \cdot H^\circ$ is right H-invariant). Moreover, if $Q = G/H$ is identified with a submanifold of G, one can just identify

$$\mathbf{Ph}\, Q = Q \cdot H^\circ. \tag{5}$$

Consider now less favorable cases, when we can still assume that G and G^* can be viewed as subgroups of the Manin group and $G \cap G^* = \{e\}$, but $G \cdot G^*$ is not a subgroup. We have in this case

$$\mathbf{Ph}\, G = \text{connected component of } \ \{e\} \ \text{ in } \ (G \cdot G^*) \cap (G^* \cdot G)$$

and instead of (4) we have

$$\mathbf{Ph}\, Q = (\mathbf{Ph}\, G \cap G \cdot H^\circ)/H = (G^* \cdot G \cap G \cdot H^\circ)/H. \tag{6}$$

If $Q = G/H$ is identified with a submanifold of G, then formula (5) is generalized to

$$\mathbf{Ph}\, Q = \mathbf{Ph}\, G \cap (Q \cdot H^\circ). \tag{7}$$

The paper is organized as follows. In Section 2 we recall the results of [1, 2] concerning the Poincaré symmetry in two dimensions. In Section 3 we consider the Euclidean symmetry in two dimensions, and in Section 4 we discuss the case of the ($SU(2)$-symmetric) Poisson 2-sphere. The above introduced notation is used without further explanation.

2 Two-dimensional Minkowski space-time

In [1] we have discussed the Poisson symmetry described by

$$G = \{(v^+, v^-, a) \colon \ v^+, v^-, a \in \mathbb{R}\}, \quad (v^+, v^-, a)(u^+, u^-, b) = (v^+ + e^a u^+, v^- + e^{-a} u^-, a + b)$$

$$H = \{(0, 0, a) \colon a \in \mathbb{R}\}, \qquad Q = G/H \cong \{(x^+, x^-, 0) \colon \ x^+, x^- \in \mathbb{R}\}$$

(x^+, x^- and v^+, v^- are light-cone coordinates), with Poisson brackets

$$\{v^+, v^-\} = \varepsilon v^+ v^-, \qquad \{a, v^\pm\} = \varepsilon v^\pm, \qquad \{x^+, x^-\} = \varepsilon x^+ x^-.$$

Then the dual Poisson group is $G^* = \{(v_+, v_-, b)\colon\ v_+, v_-, b \in \mathbb{R}\}$ with multiplication

$$(v_+, v_-, b)(u_+, u_-, b') = (v_+ e^{-\varepsilon b'/2} + e^{\varepsilon b/2} u_+, v_- e^{-\varepsilon b'/2} + e^{\varepsilon b/2} u_-, b + b'),$$

and Poisson brackets $\{v_+, v_-\} = 0$, $\{b, v_\pm\} = \pm v_\pm$. We shall denote the elements of $H^\circ = \{(\eta_+, \eta_-, 0) \in G^*\}$ simply by $\eta \equiv (\eta_+, \eta_-)$ and the elements of Q by $x \equiv (x^+, x^-)$. For $(x, \eta) \in \mathbf{Ph}\, Q$ (formula (7)), we obtain (using (9) of [1]) the groupoid projections

$$(x, \eta)_L = (x^+, x^-), \qquad (x, \eta)_R = \left(\frac{x^+}{1 + \varepsilon \eta_- x^-}, \frac{x^-}{1 - \varepsilon \eta_+ x^+} \right), \tag{8}$$

Poisson brackets (using the fact that the map $(x, \eta) = \xi \mapsto (\xi_L, \xi_R) \in Q \times \overline{Q}$ is Poisson)

$$\{x^+, x^-\} = \varepsilon x^+ x^-, \quad \{\eta_+, \eta_-\} = \varepsilon \eta_+ \eta_-, \quad \{\eta_+, x^+\} = 1 - \varepsilon \eta_+ x^+,$$

$$\{\eta_+, x^-\} = -\varepsilon \eta_+ x^-, \quad \{\eta_-, x^+\} = \varepsilon \eta_- x^+, \quad \{\eta_-, x^-\} = 1 + \varepsilon \eta_- x^-$$

and the moment map $J\colon \mathbf{Ph}\, Q \to \overline{G^*}$ (using again (9) of [1] and the change of coordinates in [2]) which associates with $(x, \eta) \in \mathbf{Ph}\, Q$ the following element of G^*:

$$\left(\eta_+ \sqrt{\frac{1 + \varepsilon \eta_- x^-}{1 - \varepsilon \eta_+ x^+}}, \eta_- \sqrt{\frac{1 + \varepsilon \eta_- x^-}{1 - \varepsilon \eta_+ x^+}}, -\frac{1}{\varepsilon} \log(1 + \varepsilon \eta_- x^-)(1 - \varepsilon \eta_+ x^+) \right).$$

Transporting the Casimir function $v_+ v_-$ on G^* to $\mathbf{Ph}\, Q$ by J we obtain the mass shell in the usual form $\eta_+ \eta_- = m^2$. Calculating the Poisson brackets of coordinates with the 'Hamiltonian' equal $\eta_+ \eta_-$ yields the following equations of motion of a free particle

$$\dot{x}^+ = \eta_-, \quad \dot{x}^- = \eta_+, \quad \dot{\eta}_+ = -\varepsilon \eta_- \eta_+^2, \quad \dot{\eta}_- = \varepsilon \eta_+ \eta_-^2.$$

It follows that the world line of the particle is a hyperbole

$$(x^+ - c^+)(x^- - c^-) = -\frac{1}{\varepsilon^2 m^2}. \tag{9}$$

Note that in $\mathbf{Ph}\, Q$ one can introduce commuting space-time coordinates as follows. Since the Poisson bivector on Q is built of commuting vector fields:

$$\pi = \varepsilon x^+ \frac{\partial}{\partial x^+} \wedge x^- \frac{\partial}{\partial x^-}, \tag{10}$$

one can realize (see [2, 7]) $\mathbf{Ph}\, Q$ in the cotangent bundle T^*Q. For the usual coordinates $(q, p) = (q^+, q^-, p_+, p_-)$ of T^*Q we have the usual canonical commutation relations, the groupoid projections from $\mathbf{Ph}\, Q$ to Q being given by

$$(q, p)_L = (q^+ e^{\frac{\varepsilon}{2} p_- q^-}, q^- e^{-\frac{\varepsilon}{2} p_+ q^+}), \qquad (q, p)_R = (q^+ e^{-\frac{\varepsilon}{2} p_- q^-}, q^- e^{\frac{\varepsilon}{2} p_+ q^+}). \tag{11}$$

Identifying projections (8) and (11) we obtain formulae relating (x, η) and (q, p). In particular, we obtain expressions for the moment map and the mass shell in the new variables:

$$J(q^+, q^-, p_+, p_-) = (P_+, P_-, \Pi_+ - \Pi_-), \qquad P_+ P_- = m^2,$$

where

$$P_+ = \frac{\sinh \frac{\varepsilon}{2} p_+ q^+}{\frac{\varepsilon}{2} q^+}, \qquad P_- = \frac{\sinh \frac{\varepsilon}{2} p_- q^-}{\frac{\varepsilon}{2} q^-}, \qquad \Pi_+ = p_+ q^+, \qquad \Pi_- = p_- q^-.$$

This gives the equations of motion $\dot{P}_\pm = 0$, $\dot{\Pi}_\pm = m^2$, which yield now world lines different from hyperboles (9):

$$q^+ = e^a \frac{\sinh \frac{\varepsilon}{2}\tau}{\frac{\varepsilon}{2} m}, \qquad q^- = e^{-a} \frac{\sinh \frac{\varepsilon}{2}(\tau - b)}{\frac{\varepsilon}{2} m},$$

where τ is the parameter and a, b are some constants.

Note that the cotangent bundle projection turns out to be a kind of geometric mean of the left and right groupoid projections:

$$x_L^+ x_R^+ = (q^+)^2, \qquad x_L^- x_R^- = (q^-)^2.$$

3 The Poisson plane

We identify the (double cover of the) Euclidean group $E(2)$ in two dimensions with the set of matrices

$$G = \left\{ \begin{pmatrix} \alpha & 0 \\ \gamma & \overline{\alpha} \end{pmatrix} \ : \quad \alpha, \gamma \in \mathbb{C}, \ \ \alpha\overline{\alpha} = 1 \right\}.$$

The standard Poisson structure (ε is the deformation parameter)

$$\{\alpha, \gamma\} = i\varepsilon\alpha\gamma, \qquad \{\gamma, \overline{\gamma}\} = 0, \tag{12}$$

corresponds to the Manin group $(SL(2,\mathbb{C}); G, G^*)$ with the scalar product defined by

$$\frac{1}{\varepsilon} \mathrm{Im}\,\mathrm{tr}\, XY \qquad \text{for } \ X, Y \in sl(2,\mathbb{C}), \tag{13}$$

and the dual Poisson group being

$$G^* = \left\{ \begin{pmatrix} \rho & n \\ 0 & \rho^{-1} \end{pmatrix} \ : \quad \rho > 0, \ n \in \mathbb{C} \right\}. \tag{14}$$

The Poisson brackets on G^* are then $\{\rho, n\} = -i\varepsilon\rho n$, $\{n, \overline{n}\} = 0$. It is convenient to use new parameters (P, s) on G^*, given by

$$\rho = \exp(\varepsilon s), \qquad n = i\varepsilon\overline{P}.$$

In the limit $\varepsilon \to 0$, these parameters become the usual translational and rotational momenta, respectively. Their Poisson brackets are simply

$$\{s, P\} = i\varepsilon P, \qquad \{P, \overline{P}\} = 0.$$

Dividing G by the Poisson subgroup H composed of diagonal matrices, we get the quotient $Q = G/H$ which may be identified with the complex plane:

$$Q \cong \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \ : \quad x \in \mathbb{C} \right\}.$$

Using (12) it is easy to find the Poisson structure induced on the plane by the projection from G on Q (given by $x = \overline{\alpha}\gamma$):

$$\{\overline{x}, x\} = 2i\varepsilon |x|^2. \tag{15}$$

If $x = x^1 + ix^2$, then $\{x^1, x^2\} = \varepsilon((x^1)^2 + (x^2)^2)$.

The subgroup H° in G^* is composed of elements of the form $(P, s) = (\eta, 0)$, where $\eta \in \mathbb{C}$. We shall denote $(\eta, 0)$ by η. Writing

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} 1 & i\varepsilon\overline{\eta} \\ 0 & 1 \end{pmatrix}$$

as a product $g^* g$, where $g \in G$, $g^* \in G^*$, we get the groupoid projections

$$(x, \eta)_L = x, \qquad (x, \eta)_R = \frac{x}{1 - i\varepsilon\overline{x}\eta}, \tag{16}$$

defined on $\mathbf{Ph}\, Q$ as given in (7), and the moment map

$$J(x, \eta) = \left(\eta \cdot \frac{|1 - i\varepsilon\overline{x}\eta|}{1 - i\varepsilon\overline{x}\eta}, -\frac{1}{\varepsilon} \log|1 - i\varepsilon\overline{x}\eta| \right). \tag{17}$$

Using the fact that $\mathbf{Ph}\, Q \ni \xi \mapsto (\xi_L, \xi_R) \in Q \times \overline{Q}$ is a Poisson map, one gets the Poisson structure of $\mathbf{Ph}\, Q$: in addition to (15) we have

$$\{\overline{\eta}, \eta\} = -2i\varepsilon|\eta|^2, \qquad \{\eta, x\} = -2i\varepsilon\eta x, \qquad \{\eta, \overline{x}\} = 2(1 - i\varepsilon\overline{x}\eta). \tag{18}$$

Now we note that the Casimir function $|P|^2$ on G^* equals $|\eta|^2$, when transported to $\mathbf{Ph}\, Q$ by the moment map. Therefore it is natural to consider the Hamiltonian

$$\mathcal{H} = \frac{1}{2}|\eta|^2$$

as describing the ('ε-analogue' of the) free dynamics. Integrating the equations of motion

$$\dot{x} = \{\mathcal{H}, x\} = \eta, \qquad \dot{\eta} = -i\varepsilon|\eta|^2\eta, \tag{19}$$

we obtain easily

$$\eta = \eta_0 \exp(-2i\varepsilon Et), \qquad x = x_0 + \eta_0 \frac{1 - \exp(-2i\varepsilon Et)}{2i\varepsilon E}, \qquad E \equiv \mathcal{H} = \frac{1}{2}|\eta_0|^2.$$

We now see the 'effect of the deformation': free trajectories are bounded. They are circles or points (never straight lines). The radius of the circle is inversely proportional to the velocity!

As in Section 1, one can introduce (somehow externally) commuting positions in $\mathbf{Ph}\,Q$, using the following representation of the Poisson bivector on Q:

$$\pi = \varepsilon|x|^2\partial_1 \wedge \partial_2 = \varepsilon(x^1\partial_1 + x^2\partial_2) \wedge (x^1\partial_2 - x^2\partial_1), \quad \text{where } \partial_j := \frac{\partial}{\partial x^j},\ j = 1, 2.$$

For $(q, p) = (q^1 + iq^2, p_1 + ip_2) \in T^*Q$ we have the usual canonical commutation relations (the only non-zero are $\{p, \overline{q}\} = 2$) and we get the following groupoid projections

$$(q, p)_L = \exp(-i\frac{\varepsilon}{2}\overline{q}p) \cdot q, \qquad (q, p)_R = \exp(i\frac{\varepsilon}{2}\overline{q}p) \cdot q. \tag{20}$$

Identifying projections (16) and (20) we obtain formulae relating (x, η) and (q, p). In particular, we obtain expressions for the moment map and the Hamiltonian in the new variables:

$$J(q, p) = (P, -\mathrm{Im}\,\overline{q}p), \qquad \mathcal{H} = \frac{1}{2}P\overline{P}, \qquad \text{where } P = \frac{\sin\frac{\varepsilon}{2}\overline{q}p}{\frac{\varepsilon}{2}\overline{q}}.$$

Since $\{P, \overline{P}\} = 0$, the 'effective' momentum P is conserved. Using $\{P, \overline{q}p\} = 2P$ and $\{q\overline{p}, \overline{q}p\} = 0$, we get $\frac{d}{dt}(\overline{q}p) = |P|^2 = 2E$ and

$$q = \frac{\sin\frac{\varepsilon}{2}q\overline{p}}{\frac{\varepsilon}{2}\overline{P}} = q_0\frac{\sin\frac{\varepsilon}{2}((q\overline{p})_0 + 2Et)}{\sin\frac{\varepsilon}{2}(q\overline{p})_0}.$$

Trajectories in terms of commuting positions q are not circles but more complicated closed curves. They arise as geometric mean of two circular motions, since

$$\xi_L \cdot \xi_R = (q, p)_L \cdot (q, p)_R = q^2 \qquad \text{for } \xi \in \mathbf{Ph}\,Q.$$

Using this property, we obtain also a remarkable formula for the Hamiltonian:

$$2\varepsilon^2\mathcal{H} = |\varepsilon P|^2 = \left|\frac{2\sin\frac{\varepsilon}{2}\overline{q}p}{q}\right|^2 = \left|\frac{\exp(i\frac{\varepsilon}{2}\overline{q}p)q - \exp(-i\frac{\varepsilon}{2}\overline{q}p)q}{q^2}\right|^2$$
$$= \left|\frac{\xi_R - \xi_L}{\xi_L\xi_R}\right|^2 = \left|\frac{1}{\xi_L} - \frac{1}{\xi_R}\right|^2.$$

4 Free motion on the Poisson sphere

We start with simple observations relating the usual free motion on the sphere $Q = G/H = SU(2)/S^1$ to the free motion on $G = SU(2)$. The Hamiltonian $H\colon T^*G \to \mathbb{R}$ of the latter may be written as

$$\mathcal{H}(\xi) = \frac{1}{2}J^2, \tag{21}$$

where $J : T^*G \to \mathfrak{g}^*$ is the moment map for the right translations ($J(\xi) = g^{-1}\xi$ for $\xi \in T^*_gG$) and $J^2(\xi)$ denotes the scalar square of $J(\xi)$ in terms of some invariant scalar product on $\mathfrak{g}^*$. Specifically, using the natural orthogonal basis

$$J_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

of $su(2)$, denoting by $\tilde{J}_k$ the corresponding left-invariant vector fields on G and (again) by J_k the associated functions on T^*G,

$$J_k(\xi) = \left\langle \xi, \tilde{J}_k(g) \right\rangle \qquad \text{for } \xi \in T^*_g G,$$

we may set

$$\mathcal{H} = \frac{1}{2}(J_1^2 + J_2^2 + J_3^2).$$

It is well known that the resulting free trajectories on $SU(2) \cong S^3$ are 'big circles' (translated one-parameter subgroups).

Let $H \cong S^1$ be the subgroup generated by J_3. To the reduction map $G \to Q = G/H$ there corresponds the symplectic reduction

$$T^*G \supset K \to T^*Q, \tag{22}$$

where the constraint set K is given by $\{J_3 = 0\}$. Since the S^1 action from the right (generated by J_3) preserves the Hamiltonian (i.e. $\{J_3, \mathcal{H}\} = 0$), projections of trajectories of H are trajectories of the projected Hamiltonian $\mathcal{H}_{\text{red}}$ on T^*Q. Moreover (since reduction (22) is implied by the configurational reduction), also configurational trajectories on $SU(2)$ are projected on configurational trajectories on $Q = S^2$.

The following lemma is easily proved.

Lemma. *Projecting all 'big circles' from $SU(2)$ to S^2, one obtains all circles on S^2.*

It is perfectly known that free trajectories on S^2 are not 'all' circles but only the 'big' ones (or points — in the case of rest). This may seem to contradict the statement before the lemma. There is no paradox, of course, because we project from T^*G only those trajectories which are in the constraint K. These are exactly those trajectories whose configurational velocity is perpendicular to $\tilde{J}_3$ (the direction of the S^1 action):

$$J_3(\xi) = 0 \iff \left\langle \xi, \tilde{J}_3(g) \right\rangle = 0 \iff v \perp \tilde{J}_3$$

($v \in T_g G$ is obtained from $\xi \in T^*_g G$ by using the invariant metric on G). One can easily check that this perpendicularity condition is exactly equivalent to the fact that the projection on S^2 is a 'big' circle (or a point).

Now we consider the standard Poisson structure on $G = SU(2)$, corresponding to the Manin group $(SL(2,\mathbb{C}); G, G^*)$, with the scalar product and the dual group as before, given by (13) and (14). In [4] we have calculated the symplectic structure of $\mathbf{Ph}\, G = SL(2,\mathbb{C})$ and solved the equations of motion for the following analogue of the free Hamiltonian (21):

$$\mathcal{H}(\xi) = \frac{1}{2}\operatorname{tr} \xi^\dagger \xi, \qquad \xi \in SL(2,\mathbb{C}).$$

We shall write the result for the rescaled Hamiltonian

$$\tilde{\mathcal{H}} := \frac{\mathcal{H} - 1}{4\varepsilon^2}, \tag{23}$$

because this one tends to (21) when $\varepsilon \to 0$, while using the following parametrization of G^*:

$$G^* \ni b = \begin{pmatrix} \rho & n \\ 0 & \rho^{-1} \end{pmatrix} = \begin{pmatrix} \exp(\varepsilon s) & 2\varepsilon w \\ 0 & \exp(-\varepsilon s) \end{pmatrix}.$$

The result is that the phase trajectory $t \mapsto \xi = gb \in G \cdot G^* = SL(2,\mathbb{C})$ satisfies

$$g^{-1}\dot{g} = \mathcal{F}(b) := \frac{i}{2}\begin{pmatrix} (2\varepsilon)^{-1}\sinh 2\varepsilon s \, + \varepsilon|w|^2 & w\exp(-\varepsilon s) \\ \overline{w}\exp(-\varepsilon s) & -(2\varepsilon)^{-1}\sinh 2\varepsilon s \, - \varepsilon|w|^2 \end{pmatrix}$$

and $\dot{b} = 0$. This means that the left groupoid projection $t \mapsto g(t)$ of the phase trajectory is the usual 'big' circle, and the motion is uniform with constant velocity given by the deformed 'Legendre transformation' $G^* \ni b \mapsto v := \mathcal{F}(b) \in \mathfrak{g}$.

The free Hamiltonian (23) is a Casimir function on G^* and commutes with left and right momenta (2). In particular, it commutes with the constraint (1), and, consequently, may be projected down to the reduced space $\mathbf{Ph}\, Q = \mathbf{Ph}\,(G/H)$. It is this function which we consider as the ε-analogue of the Hamiltonian of the free motion on S^2. Note that the resulting trajectories on S^2 being left projection of trajectories in $\mathbf{Ph}\, Q$, are at the same time the projections of those 'big' circles on $SU(2)$ which come from phase trajectories living in $K = G \cdot H^\circ$ (formula (3)). This follows from the fundamental property of morphisms of groupoids [6] (morphisms commute with groupoid projections). Phase trajectory lives in K if and only if $b \in H^\circ$, i.e.

$$b = \begin{pmatrix} 1 & 2\varepsilon w \\ 0 & 1 \end{pmatrix}.$$

For such b,

$$\mathcal{F}(b) = \frac{i}{2}\begin{pmatrix} \varepsilon|w|^2 & w \\ \overline{w} & -\varepsilon|w|^2 \end{pmatrix}$$

has vanishing scalar product with J_3 only when $w = 0$. It follows that the velocity is perpendicular to the S^1 action only if it is zero. This means that we never get a 'big' circle on S^2. Moreover, since velocities may have here any angle (different from 0 and $\pi/2$) with J_3, we get all circles on S^2, except the big ones. This fact corresponds to the previous result concerning the Poisson plane: trajectories could be any circles 'except' the straight lines.

References

[1] S. Zakrzewski, "Poisson space-time symmetry and corresponding elementary systems", in: *Quantum Symmetries*, Proceedings of the II International Wigner Symposium, Goslar 1991, H.D. Doebner and V.K. Dobrev (Eds.), pp. 111–123.

[2] S. Zakrzewski, "Poisson Poincaré particle and canonical variables", in: *Generalized Symmetries*, Proceedings of the International Symposium on Mathematical Physics, Clausthal, July 27–29, 1993, H.-D. Doebner, V.K. Dobrev and A.G. Ushveridze (Eds.), 1994, pp. 165–171.

[3] S. Zakrzewski, "On the classical κ-particle", in: *Quantum Groups, Formalism and Applications*, Proceedings of the XXX Winter School on Theoretical Physics 14–26 February 1994, Karpacz, J. Lukierski, Z. Popowicz, J. Sobczyk (eds.), Polish Scientific Publishers PWN, Warsaw 1995, pp. 573–577. Also: hep-th/9412098.

[4] S. Zakrzewski, "Classical mechanical systems based on Poisson symmetry", in: Proceedings of the II German-Polish Symposium, Zakopane, September 1995, *Acta Physica Polonica B*, vol. **27** No. 10 (1996), 2801–2810.

[5] S. Zakrzewski, "Phase spaces related to standard classical r-matrices", *J. Phys. A*: Math. Gen. **29** (1996) 1841–1857.

[6] S. Zakrzewski, "Quantum and classical pseudogroups. Part I and II", *Comm. Math. Phys.* **134** (1990), 347–395.

[7] S. Zakrzewski, "Geometric quantization of Poisson groups — diagonal and soft deformations", Proceedings of the Taniguchi Symposium *Symplectic geometry and quantization problems*, Sanda (1993), Y. Maeda, H. Omori and A. Weinstein (Eds.), Contemporary Mathematics **179**, 1994, 271–285.